Ricci flow Lecture Note

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Preface When I was a sophomore, first learned **Ricci flow** in 2024's Summer School in Geometry in USTC. This lecture note is sorted out from a lecture called "Some aspect of Ricci flow on non-compact manifolds" by Prof. Eric Chen in UCLA. Because of my terrible method in Riemannian geometry and PDE ☹, I just understand little part in this lecture, and most of notes are copied by the blackboard, if you find any faults in this note(obviously☻)...That's all my problem.

Rmk: The important ideas for me at this stage are edited in red font, and the ones I can't verify up to now are written by gray, which I just copy the blackboard(the whole part of Lec3 is copied from board).

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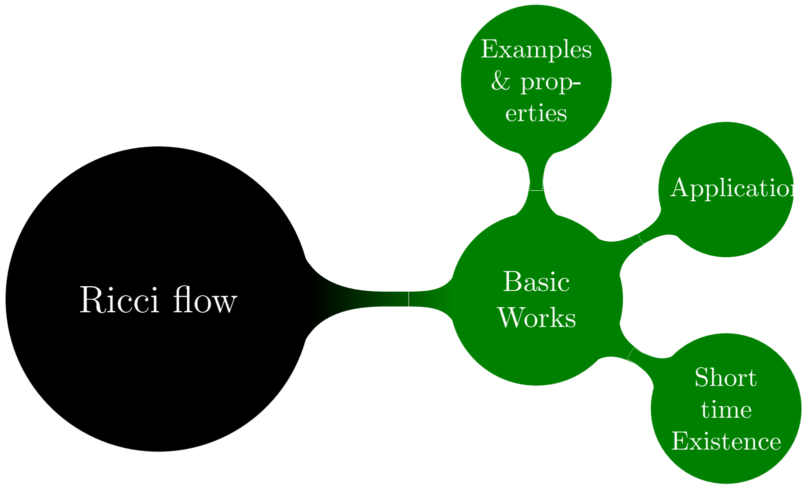

1.1 Some basic concepts in Riemannian Geometry

¶ review Riemannian Geometry First, we recall some basic concepts in Riemannian Geometry, one can see more details in [15, 20, 26], or review some brief notations in [30, 12]. Let (M, g) a Riemannian manifold with metric g, Sometimes we ignore g and just denote M.

† Levi-civita connection A $Levi-civita\ connection$ on any vector field of M ∇ is a connection

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

s.t.

1. $torsion-free$: $\nabla_X Y - \nabla_Y X = [X, Y]$
2. $metric-compatibility$: $\nabla g = 0$

† Riemannian curvature tensor Denote $R(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{[X,Y]}$, and a *Riemannian curvature tensor* is a $(0,4)-$type tensor

$$\mathrm{Rm}(X, Y, Z, W) = R_{ijkl} X^i Y^j Z^k W^l$$

where $\mathrm{Rm}(X, Y, Z, W) :=< R(X, Y)Z, W >$.

† sectional curvature A *sectional curvature* on $X \wedge Y$ is defined by

$$K(X, Y) := \frac{R(X, Y, X, Y)}{< X, X >< Y, Y > - < X, Y >^2}$$

† Ricci curvature & scalar curvature A *Ricci curvature* is defined by

$$\mathrm{Ric}(X, Y) := \mathrm{tr}\mathrm{Rm}(X, -, Y, -)$$

and *scalar curvature*

$$R := \mathrm{tr}\mathrm{Ric}$$

¶ **curvature & topology**

† **Question 1** What does curvature tell us about topology?

Let M a m dimensional Riemannian Manifold with the Riemannian Metric g, then we have

- If sectional curvature $K \equiv \text{const}$, then

  1. $M \cong S^m/\Gamma$ or
  2. $M \cong \mathbb{R}^m/\Gamma$ or
  3. $M \cong \mathbb{H}^m/\Gamma$,

  where $\Gamma \in \text{Isom(M)}$, one can see more ditails in [6].

- If Ricci curvature $Ric = \lambda g$, where $\lambda \equiv \text{const}$ and $m \geq 3$, we call M a *Einstein manifold* in this case. We shall show some basic properties of such manifold under the RF in subsection1.3.

- If scalar curvature $R \equiv \text{const}$, which links to the *Yamabe Problem*[1].

conjecture 1 (Yamabe,1960). [2] *Let M a closed Riemannian manifold of dimension $m \geq 3$, is there a metric $\tilde{g}$ conformal to g that has constant scalar curvature R?*

† **Question 2** Given M, can deform g to $\tilde{g}$ with a nicer curvature?

If $m = 2$, we say this question is solved by *uniformization*, and we shall show it in the subsection1.4.

1.2 Ricci flow Motivation

¶ **definition**

definition 1.1 (Hamilton,1982). Let M a Riemannian manifold, a *Ricci flow*[3] is a family of metrics $g(t)$ in M^m, with $t \in I \subset \mathbb{R}$, s.t.

$$\partial_t g = -2\text{Ric(g)}$$

¶ **harmonic coordinates**

definition 1.2. Let M a Riemannian manifold, a coordinate chart (U, φ) with parameters $(x^1, \cdots, x^m)$ is *harmonic*, if $\triangle_g x^i = 0, i = 1, \cdots, m$.

remark 1.1. *We have RF*

$$\text{Ric}_{ij} = -\frac{1}{2} \triangle g_{ij} + \text{l.o.t}$$

under the harmonic coordinate.

[1] How to find a nice metric s.t. $R \equiv \text{const}$.

[2] This problem has been solved by R.Schoen in 1984, one can see [28].

[3] denoted by RF.

¶ **Einstein-Hilbert functional**

definition 1.3. We say

$$S(g) = \frac{\int_M R dV_g}{\mathrm{Vol}(g)^{\frac{n-2}{n}}}$$

is an $Einstein-Hilbert\ functional$.

One can compute the gradient of S

$$\mathrm{grad} S = \mathrm{Vol}(g)(-\mathrm{Ric} + \frac{1}{2}g - \frac{n-2}{2n}\overline{R})$$

where $\overline{R} = \int R/\mathrm{Vol}(g)$.

¶ **heunstic picture**

† $m = 2$ **dimensions** In 2 dimensions, we know that Ricci curvature can be written in terms of *Gauss curvature* K as $\mathrm{Ric}(g) = Kg$. Working directly from the equation

$$\partial_t g = -2Kg$$

, we can see that regions in where $K < 0$ tend to **expand**, and regions where $K > 0$ tend to **shrink**. One can learn more details in [30], it's no hard to guess that the Ricci flow tends to make a S^2 "rounder". This is indeed the case, and there is an excellent theory which show that Ricci flow on any **closed surface** tends to make the Gauss curvature constant, after renormalization. One can read more of this dimension in [13].

† $m \geq 3$ **dimensions** However, we will encounter **singularities** at the **neck pinch** in dimension over 3, one can see more about higher dimension of spheres in [1].

1.3 Some Examples and Properties of RF

¶ **Einstein manifold** For a Einstein manifold (M, g), we have $\mathrm{Ric(g_0)} = \lambda g_0$, for some $\lambda \in \mathbb{R}$, then the solution of RF(Let $g(0) = g_0$) is

$$g(t) = \begin{cases} 1 - 2\lambda t g_0, & t > 0 \quad \text{if} \lambda < 0 \\ g_0, & t \in \mathbb{R} \quad \text{if} \lambda = 0 \\ 1 - 2\lambda t g_0, & t < 0 \quad \text{if} \lambda > 0 \end{cases}$$

example 1.1. Let S^n with a round metric g_0, we have $\mathrm{Ric}(g_0) = (n-1)g_0$. Then the solution of RF is

$$g(t) = (1 - 2(n-1)t)g_0, \quad t \leq T = \frac{1}{2(n-1)}$$

One can see the sphere collapses to a point at time $T = \frac{1}{2(n-1)}$. If we choose a hyperbolic metric g_0, then $Ric(g_0) = 1(n-1)g_0$ and $g(t) = (1+2(n-1)t)g_0$, so the manifold expands for all time.

One can link this example to subsection1.2 and it(the Einstein manifold) is a special case of Ricci solitons which we will show in the following paragraphs.

¶ **product & quotient**

- If $(M, g_M(t))$ and $(N, g_N(t))$ with $t \in I$ are RF, then so is $(M \times N, g_M(t) \oplus g_N(t))$
- If $\varphi : M \to M$ is an isometric of (M, g_0) and $(M, g(t))$ with $t \in [0, T]$ is a solution of RF, then φ also isometric of $g(t)$. So quotient M/Γ is also presevered by RF.

¶ **invariance properties**

- **time translation**: If $(M, g(t))$ is a RF, $a \in \mathbb{R}$, then $(M, g(t+a))$ is also a RF.
- **parabolic rescaling**: If $(M, g(t))$ is a RF, then $(M, \lambda^2 g(\lambda^{-2} t))$ is also a RF.
- **diffeomorphism invariance**: If $(M, g(t))$ is a RF, $\phi : M \to M$ is a diffeomrophism, then $(M, \phi^* g(t))$ is a RF.

¶ **Ricci soliton** There is a more general notion of self-similar solution than the uniformly shrinking of expanding solutions. To understand such solutions, we must consider the idea of modifying a flow by a family of diffeomorphisms. Let $X(t)$ a time dependent vector fields on M, generating a family of diffeomorphism ϕ_t.

Let $Y \in \Gamma(TM), \lambda \equiv \text{const}$. We can find a solution of RF

$$-2\mathrm{Ric}(\hat{g}) = \partial_t \hat{g}$$

, where $\hat{g} = (1 - 2\lambda t) g$ to be

$$-2\mathrm{Ric}(g_0) = L_Y g_0 - 2\lambda g_0$$

where $g(t) = g_0$. One can verify this solution by following proposition(Let $\sigma(t) = 1 - 2\lambda t$ $X = \frac{1}{\sigma(t)} Y$).

proposition 1.1. *Let $\sigma(t) \in C^\infty(M)$, and define*

$$\hat{g}(t) := \sigma(t) \phi_t^*(g(t))$$

then we have

$$\partial_t \hat{g} = \sigma'(t) \phi_t^*(g(t)) + \sigma(t) \phi_t^*(\partial_t g) + \sigma(t) \phi_t^*(L_X g)$$

definition 1.4. We call $\hat{g} := (1 - 2\lambda t) g_0$ above the *Ricci soliton*. Moreover, such solitons called $expand, shrink, steady$, when $\lambda < 0, > 0, = 0$.

example 1.2 (Einstein manifold)**.** Let M be any Einstein manifold and $Y \equiv 0$.

example 1.3 (Gauss soliton)**.** Let M be $(\mathbb{R}^m, g^0)$ and $Y = \pm s \partial_s, \lambda = \pm 1$.

example 1.4 (Hamilton's cigar soliton)**.** Let M be $(\mathbb{R}^2, \delta_{ij}/1 + s^2)$ and $Y = \partial_s, \lambda = 0$.

One can learn more details in [30, 12].

1.4 Survey of Applications

¶ classify compact manifold of dimension m

- $m = 2$: independent proof of $uniformization$, one can see [32, 11, 10]. We also can link this theorem to Riemann's work in Complex Analysis, such that every surfaces have a conformal metric of constant Gaussian curvature, or ref to Chern's work in Differential Geometry.
- $m = 3$: $Thurstons$ $geometrization$, one of the special cases of it is $Poincare$ *conjugate* and one can see [25].
- $m \geq 4$: understand singularities of RF.
- $m \geq 5$: do more surgeries.

theorem 1.1 (uniformization). *Let M be a closed Riemannian manifold of dimension $m = 2$, then it is conformally equivalent to one of constant sectional curvature.*

conjecture 2 (Thurstons geometrization). *Every 3 dimensional manifolds admits a geometric decomposition.*

One can see more details about geometric decomposition in [30].

¶ curvature pinching

- Any compact manifold (M^3, g) with Ric > 0 is diffeomorphic to S^3/Γ. It is the well-known consequence of Hamilton in 1982, one can see [18].
- Any compact manifold (M^m, g) with Rm > 0 is diffeomorphic to S^m/Γ, one can see [5].
- Any compact manifold (M^m, g) with sectional curvature s.t. $K \in (1/4, 1]$, then diffeomorphic to S^m/Γ, one can see [8, 7].
- stability or instability near fixed points (Einstein metric/solitons).

¶ other applications

- Pinching in noncompact settings.
- With conored at ∞, RF behavior on noncompact manifolds(AE,AC,cylinder).
- $Kahler$ setting of dimension 2.

1.5 Short time Existence

† **Question** Can we start RF to use it?

One can learn some basic parabolic theory in euclidean space or manifold in [30].

¶ **parabolic theory-linear** We assume M to be closed.

definition 1.5. Let $E \to M$ a vector bundle of M, $L:\Gamma(E)\to\Gamma(E)$ is a *linear second order differential operator*(*linear C^2 operator*), if in local coordinate $\{x^i\}$ on M and local frame $\{e_\alpha\}$ on E, we have

$$L(v) = [a^{ij}_{\alpha\beta}\partial_i\partial_j v^\beta + b^i_{\alpha\beta}\partial_i v^\beta + c_{\alpha\beta}v^\beta]e_\alpha$$

for $v = v^\alpha e_\alpha$

definition 1.6. *Principal symbol* of L is $\sigma(L) : \tilde{\pi}^*E \to \tilde{\pi}^*E$[4], where $\tilde{\pi} : T^*M \to M$ denote a bundle projection, given by

$$\sigma(L)(x,\xi)v := (a^{ij}_{\alpha\beta}\xi_i\xi_j v^\beta)e_\alpha \in E_x$$

for $x \in M, \xi \in T^*_x M$.

definition 1.7. The equation

$$\partial_t v = L(v), \quad v \in \Gamma(E)$$

is *strongly parabolic* if

$$< \sigma(L)(x,\xi)v, v >_E \geq \lambda|\xi|^2|v|^2_E, \quad \text{some } \lambda > 0$$

example 1.5 (motivation of $\partial_t g = -2\mathrm{Ric}(g), \quad g,h \in \Gamma(\mathrm{Sym}^2 T^*M)$[5]). If

$$Lh_{pq} = \triangle h_{pq} = g^{ij}\nabla_i\nabla_j h_{pq} = g^{ij}\partial_i\partial_j + \text{l.o.t}$$

then $\sigma(L)(x,\xi)h = g^{ij}\xi_i\xi_j h$. So

$$< \sigma(L)(x,\xi)h, h >= |\xi|^2|h|^2$$

Thus $\partial_t h = \triangle h$ is strongly parabolic.

¶ **parabolic theory-nonlinear**

definition 1.8. The *nonlinear* PDE

$$\partial_t v = P(v), \quad P:\Gamma(E)\to\Gamma(E)$$

such as $g \mapsto -2\mathrm{Ric}(g)$, given by(in local coordinate $\{x^i\}$ on M and local frame $\{e_\alpha\}$ on E)

$$P(v) = [a^{ij}_{\alpha\beta}(x,v,\nabla v)\partial_i\partial_j v^\beta + b^\alpha(x,v,\nabla v)]e_\alpha$$

is *strongly parabolic*, if the linear one at $w \in \Gamma(E)$

$$\partial_t v = [DP(w)]v$$

is strongly parabolic.

A well-known theorem is:

theorem 1.2. *Let M a smooth manifold, $E \to M$ a vector bundle, and $P:\Gamma(E)\to\Gamma(E)$ s.t. $\partial_t v = P(v)$ is strongly parabolic at $w \in \Gamma(E)$. Then*

$$\begin{cases} \partial_t v = P(v) \\ v(0) = w \end{cases}$$

has a ***unique*** *smooth solution for $t \in [0,\epsilon)$ (*some ϵ small*).*

[4]Note that $\tilde{\pi}^*E$ is a vector bundle over T^*M whose fiber at $(x,\xi) \in T^*M$ is E_x.

[5]Let $E = \Gamma(\mathrm{Sym}^2 T^*M)$

¶ **linearization of** Ric   Recall

$$\partial_t g = -2\mathrm{Ric}(g)$$

where $g, \mathrm{Ric}(g) \in \Gamma(\mathrm{Sym}^2 T^* M)$

Unfortunately, as we have shown,

proposition 1.2. *The linearization of* $-2\mathrm{Ric}$ *is*

$$[D(-2\mathrm{Ric})]h_{ij} = g^{pq}\nabla_p\nabla_q h_{ij} - L_X g + \mathrm{l.o.t}$$

*, but which is **NOT** strictly parabolic, where* $X = (\mathrm{div}h - \frac{1}{2}\nabla\mathrm{tr}h)_i = \nabla^j h_{ij} - \frac{1}{2}\nabla_i g^{kl} h_{kl}$.

¶ **DeTurck trick** [6] To resolve weak parabolicity, recall $[D(-2\mathrm{Ric})]h_{ij} = \triangle h_{ij} - L_X g + \mathrm{l.o.t}$, where $X = \mathrm{div}h - \frac{1}{2}\nabla\mathrm{tr}h$. We fix a $\overline{g}$ to be background metric, define

$$W^k = W(t)^k := g(t)^{pq}(\Gamma_{g\,pq}^{\ k} - \overline{\Gamma}_{\overline{g}\,pq}^{\ k})$$

and$P : \Gamma(\mathrm{Sym}^2 T^* M) \to \Gamma(\mathrm{Sym}^2 T^* M)$ by

$$g \mapsto L_W g$$

this has *linearization*

$$[DP(g)]h_{jk} = L_X g + \mathrm{l.o.t}$$

where $X = \mathrm{div}h = \frac{1}{2}\nabla\mathrm{tr}h$, So

$$\partial_t g = -2\mathrm{Ric}(g) + P(g) = -2\mathrm{Ric}(L_W g)$$

which is **Ricci-DeTurck** flow.

† **Question**   How to go back to RF? By the DeTurck trick, we get one of Hamilton's works in 1982, see[18].

proposition 1.3. *If* $\phi_t : M \to M$ *solving*

$$\begin{cases} \partial_t \phi_t(p) = -W(\phi(p), t) \\ \phi_0 = id \end{cases}$$

and $g(t)$ *solves* $\mathrm{Ricci-DeTurck}$ *flow, then* $\tilde{g}(t) := \phi_t^* g(t)$ *is a solution of RF,*

$$\begin{cases} \partial_t \tilde{g} = -2\mathrm{Ric}(\tilde{g}) \\ \tilde{g}(0) = g_0 \end{cases}$$

, then

Finally, we get two consequences.

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[6]One can learn some Specific description of this trick and the proof of "existence-unique" in [30].

theorem 1.3 (Short-time existence)**.** *Given a compact M with a smooth background metric g_0, there exists $\epsilon > 0$ and a Ricci flow $g(t)$ with $t \in [0, \epsilon)$ s.t.*

$$\begin{cases} \partial_t g = -2\mathrm{Ric}(g) \\ g(0) = g_0 \end{cases}$$

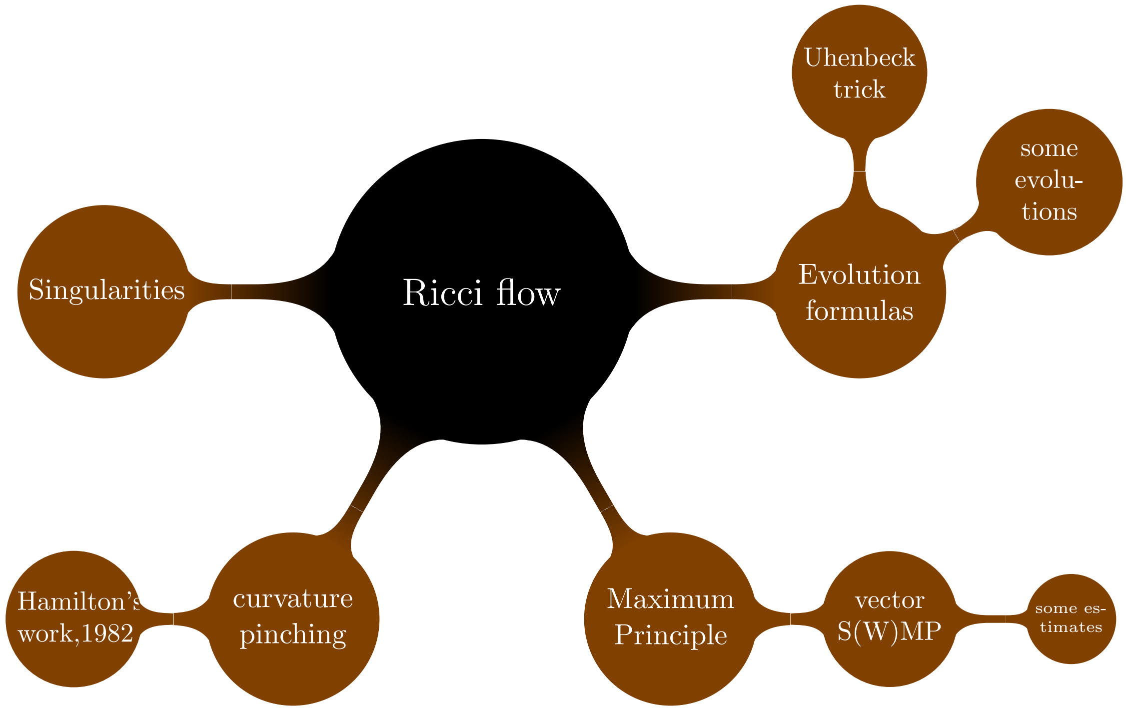
theorem 1.4 (uniqueniss of RF)**.** *The RF starting from a compact M is unique.*

remark 1.2. *But in long time we may encounter some singularities as we have shown in the former subsections.*

¶ noncompact cases

If M is complete but noncompact with $|Rm| \leq K$, then short-time existence. ...

2 Lecture 2



2.1 Evolution formulas

† **Question** How does M change under RF?

¶ **Uhlenbeck trick** We want to find a nice frame to compute. Recall if $\partial_t g_{ij} = h_{ij}$, then $\partial_t \Gamma_{ij}^k = \frac{1}{2} g^{kl}(\nabla_i h_{lj} + \cdots)$. Observe that $(M, g(t))$ with $t_1 \in [0, T)$ is a Ricci flow and since $\frac{d}{dt} e_i(t) = \mathrm{Ric}(e_i(t))$. Then note

$$\frac{d}{dt} g_t(e_i(t), e_j(t)) = -2\mathrm{Ric}(e_i t, e_j(t)) + \mathrm{Ric}(e_i(t), e_j(t)) + \mathrm{Ric}(e_i(t), e_j(t)) = 0$$

fixed metric with time-dependent frame.

definition 2.1. Let $\pi : M \times I \to M$[7] and $T^{spat}(M \times I) = \pi^* TM \subset T(M \times I)$. Define a **connection**

$$\tilde{\nabla} : \Gamma(T^{spat}(M \times I)) \to \Gamma(T^*(M \times I)) \otimes \Gamma(T^{spat}(M \times I))$$

by

$$\tilde{\nabla}_V X := \nabla_V X \quad \text{for } V \in \Gamma(T^{spat}(M \times I))$$

and

$$\tilde{\nabla}_{\partial_t} X := \partial_t X - \mathrm{Ric}_t(X)$$

remark 2.1 (metric compatibility). *For $X, Y \in \Gamma(T^{spat}(M \times I))$, we have*

$$\tilde{\nabla}_{\partial_t}(g_t(X, Y)) = g(\tilde{\nabla}_{\partial_t} X, Y) + g(X, \tilde{\nabla}_{\partial_t} Y)$$

[7] $[0, T) = I$

i.e. $\nabla_{\partial_t} g_t = 0$. *For example, if* X, Y *are independent of time, then*

$$\tilde{\nabla}_{\partial_t}(X,Y) = 0$$

The musical isomorphism $\flat, \sharp$ behave well with $\tilde{\nabla}$.

corollary 2.1. *If* $X \in \Gamma(T^{spat}(M \times I))$ *and* Y *a stationary vector field, then*

$$(\tilde{\nabla}_{\partial_t} X^\flat)(Y) = (\tilde{\nabla}_{\partial_t} X)^\flat(Y)$$

and similarly for $\sharp$.

¶ evolution of volume under RF Note $\tilde{\nabla}_{\partial_t} g = 0$, then we have $\tilde{\nabla}_{\partial_t} dV_g = 0$. Fix stationary on basis at $t_0 \in I$, note then that

$$\begin{aligned} 0 &= (\tilde{\nabla}_{\partial_t} dV_g)(e_1, \cdots, e_n) \\ &= \partial_t(dV_g(e_1, \cdots, e_n)) - \sum_{i=1}^n dV_g(e_1, \cdots, \tilde{\nabla}_T e_i, \cdots, e_n) \\ &= \partial_t(dV_g(e_1, \cdots, e_n)) + \sum_{i=1}^n \mathrm{Ric}(e_i, e_i) dV_g(e_1, \cdots, e_n) \end{aligned}$$

So

$$\partial_t \int_M dV_g = -\int R dV_g$$

remark 2.2. Volume – normallized *Ricci flow*

$$\tilde{g}(t) = \mathrm{Vol}(g(t))^{-2/n} g(t)$$

then

$$\partial_t \tilde{g} = -2\mathrm{Ric} + 2\frac{\overline{R}}{n}$$

¶ evolution of Rm Our goal is to compute $\partial_t \mathrm{Rm}$. First we have

$$\tilde{\nabla}_{\partial_t} \mathrm{Rm} = \partial_t \mathrm{Rm} + \mathrm{Ric} * \mathrm{Rm}$$

and note that

$$\tilde{\nabla}_{\partial_t} \mathrm{Rm}(X,Y) + \tilde{\nabla}_X \tilde{\mathrm{Rm}}(Y,T) + \tilde{\nabla}_Y \mathrm{Rm}(T,X) = 0$$

Then fix X, Y, Z static vector field commuting pairwise with ∂_t and **parallel** at (p_0, t_0), i.e. $\nabla_X = \nabla_Y = \nabla_Z = 0$.

One can see more details in [30] and we just show the consequence:

1.
$$\tilde{\mathrm{Rm}}(\partial_t, X)Y = -\sum_{i=1}^n (\nabla_{e^i} \mathrm{Rm}(X, e_i)Y)$$

2.
$$\tilde{\nabla}_{\partial_t} \mathrm{Rm} = \triangle \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm}$$

then

$$\partial_t \mathrm{Rm} = \triangle \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm}$$

¶ **evolution of** Ric **&** R Recall $\tilde{\nabla}_{\partial_t}$ is metric compatible, then

$$\tilde{\nabla}_{\partial_t}\mathrm{Ric} = \triangle \mathrm{Ric} + \mathrm{Rm} * \mathrm{Ric}$$

i.e.

$$\partial_t \mathrm{Ric} = \triangle \mathrm{Ric} + \mathrm{Rm} * \mathrm{Ric}$$

trace again, we have

$$\partial_t R = \triangle R + 2|\mathrm{Ric}|^2$$

¶ **evolution of derivatives of** Rm

1. $$\tilde{\nabla}_{\partial_t}\nabla\mathrm{Rm} = \triangle\nabla\mathrm{Rm} + \nabla\mathrm{Rm} * \mathrm{Rm}$$

2. in general,

$$\partial_t|\nabla^k\mathrm{Rm}|^2 \le \triangle|\nabla^k\mathrm{Rm}|^2 - 2|\nabla^{k+1}\mathrm{Rm}|^2 + C_{k,n}\sum_{i+j=k}\mathrm{Rm}|\nabla^j\mathrm{Rm}||\nabla^k\mathrm{Rm}|$$

2.2 Maximum Principle and Applications

† **Question**   What do the evolution equations for Rm, Ric, *etc.* tell us about their behavior under RF?

¶ **maximum principle**

theorem 2.1 (scalar weak maximum principle)**.** *Let M be compact, with a family of metrics $\{g(t)\}_{t\in[0,T)}$ and $u \in C^\infty(M\times[0,T)])$ s.t.*

$$\partial_t u \le \triangle u + X_t\nabla u + f(u,t)$$

where X_t is a smooth vector field and f is a smooth function, with $u \le \phi$ on $M\times\{0\}$ and $\partial_t\phi \ge f(\phi(t),t)$, $\phi \in C^\infty([0,T)])$. Then

$$u(-,t) \le \phi(t)$$

everywhere.

theorem 2.2 (strong maximum principle)**.** *If M is connected, $u(x,t)\le\phi(t)$ on $M\times[0,T]$ and $u(x_0,T)=\phi(T)$, then $u(x,t)=\phi(t)$ on $M\times[0,T]$.*

¶ **application of maximum principle(on** R**)**   Let $E = \mathrm{Ric} - \frac{R}{n}g$, we have

$$\partial_t R = \triangle R + 2|\mathrm{Ric}|^2 = \triangle R + \frac{2}{n}R^2 + 2|E|^2] \ge \triangle R + \frac{2}{n}R^2$$

If $g(t)$ a RF, $R \ge R_0$ at $t=0$, then

$$\phi(t) = \frac{1}{\frac{1}{R_0} - \frac{2}{n}t}, \quad \partial_t\phi = \frac{2}{n}\phi^2, \quad \phi(0) = R_0$$

Since weak maximum principle, we have

$$R(x,t) \ge \phi(t) \ge R_0$$

corollary 2.2. *Following are corollaries:*

- $R(-,0) \geq R_0$, *sufficiently small,* $R \geq -\frac{n}{2t}$*;*
- ancient *flow, we have* $R \geq 0$*;*
- *If* $R(-,0) \geq R_0 > 0$*, then* $T < \frac{n}{R_0}$*;*
- immortal *flow,* $t \geq 0$*, we have* $\min R(-,t) \leq 0$*;*
- eternal *flow, we have Ricci flat for compact ones.*

¶ curvature/derivative estimates

theorem 2.3. *Let* $(M, g(t))$ *a compact RF with* $|\mathrm{Rm}| \leq K$ *on* $[0,T)$*, then*

$$|\nabla^k \mathrm{Rm}| \leq \frac{ClK}{t^{l/2}}, \quad t \in [0, \frac{1}{K}]$$

¶ long-time existence criterion

theorem 2.4. *If* $(M, g(t))_{t\in[0,T)}$ *is a compact RF, T maximal, then*

$$\lim_{t\to T} \|R_m\|_\infty = +\infty$$

2.3 Curvature Pinching

† **Question** How does RF improve in long time cases?

¶ vector-valued maximum principle We assume that M to be compact, and $g(t)$ be arbitrary family of Riemannian metrics.

1. $E \to M \times [0,T)$ a *Euclidean* vector bundle with metric compatible connection ∇, denote $lift$ of ∂_t is ∇_{∂_t}.
2. $C \subset E$ is a *subbundle*, $C_{x,t} = C \cap E_{x,t} \subset E_{x,t}$ is parallel in spatial direction, i.e. if $\gamma(s)$ spatial, $e(0) \in E_{\gamma(0),t}$ and $\nabla_{\dot\gamma(s)} e(s) = 0$, then $e(s) \in C_{\gamma(s),t}$.
3. Φ is a smooth vector field on each fiber $E_{x,t}$ s.t. flow of $\nabla_{\partial_t} + \varphi$ preserves C.
4. $u \in C^\infty(M \times [0,T); E)$ s.t. $\nabla_{\partial_t u} = \triangle u + \varphi(u)$.

theorem 2.5 (vector WMP)**.** *In above setting, if* u *takes value only in* C *on* $M \times \{0\}$*, then* u *takes values only in* C *throughout* $M \times [0,T)$*.*

example 2.1. Fixed g and E a trivial bundle. $\partial_t u = \triangle u + f(u)$, and $u(-,0) \leq \varphi(0), \partial_t \varphi \geq f(\varphi)$. Let $\Phi = f$, $C_{x,t} = [f(t), \infty)$, then we get scalar WMP.

example 2.2. $g(t)$ is a RF, ∇ from Uhlenbeck trick

$$\nabla_{\partial_t} \mathrm{Ric} = \triangle \mathrm{Ric} + Q(\mathrm{Ric}, \mathrm{Rm})$$

when $m = 3$, $\mathrm{Rm} \cong \mathrm{Ric}$.

theorem 2.6 (vector SMP)**.** *Same setting, if* u *takes values in* C*,* $u(x_0,t_0) \in \partial C_{x_0,t_0}$ *at some* (x_0,t_0) *then* u *only takes values in* ∂C *on* $M \times [0,t_0]$*.*

¶ **application** Now we show the consequence of Hamilton in 1982, see [18].

theorem 2.7. *If $(M, g(t))_{t\in[0,T)}$ a compact RF and $\text{Ric}(g(0)) \geq 0$, then $\text{Ric}(g(t)) \geq 0$ for all $t \geq 0$, and either*

- $\text{Ric}(g(t)) > 0$ *for all* $t > 0$;
- $(M, g(t))$ *flat;*
- M *is a quotient of* $N^2 \times \mathbb{R}$, N *is a* $2-$ *sphere.*

remark 2.3. *In fact, for* $\text{Ric} = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ *with* $\lambda_1 \leq \lambda_2 \leq \lambda_3$ *and* $\epsilon \in [0,1]$, *then*

$$\{\text{Ric} : \lambda_1 \geq \epsilon\lambda_3 \geq 0\}$$

is preserved by RF and

$$\{\text{Ric} : \lambda_3 - \lambda_1 \leq (\lambda_1 + \lambda_2 + \lambda_3)^{1-\delta}\}$$

preserved.

We note that

$$0 \leq 1 - \frac{\lambda_1}{\lambda_3} \leq 3\lambda_3^{\delta}$$

the eigenvalues pinched when curvature large, after showing

$$\frac{R_{\max}}{R_{\min}} \to 1 \quad \text{as } t \to T \text{ (when } \text{Ric}(g_0)) > 0$$

Then we have(see [18]):

theorem 2.8 (Hamilton,1982)**.** *If (M^3, g_0) is a compact one with $\text{Ric}(g_0) > 0$, then*

$$M^3 \cong S^3/\Gamma$$

remark 2.4. *In higher dimension we need other pinching. The preserved conditions are:*

- $$\text{Rm} = W + E \wedge g + Rg \wedge g$$

  where (M, g) s.t. $|\text{Rm}| < \epsilon_n R$, then $M \cong S^n/\Gamma$, for $n \geq 4$.

- $$\text{Rm} : \bigwedge^2 T^*M \to \bigwedge^2 T^*M$$

  where $\text{Rm} \geq 0$(imply sectional curvature $K \geq 0$) is preserved under RF in any dimension....

- PIC:*positive isotropic curvature. Let $\wedge^2_{\mathbb{C}} T^*M$, $\text{Rm}(\omega, \overline{\omega}) \geq 0$ for certain ω, then*

  $$\text{WPIC2} \Longrightarrow \text{WPIC1} \Longrightarrow \text{WPIC}$$

  Imply that

  $$\text{sec.K} \geq 0 \Longrightarrow \text{Ric} \geq 0 \Longrightarrow R \geq 0$$

  And

  $$\text{PIC1} > 0 \Longrightarrow \textit{converge to round } S^n/\Gamma$$

3 Lecture 3

3.1 Singularity behavior

Recall that sometimes (normalized) RF converges.

† **Question** What happens/ what do we "neckpinch" do at nontrivial singularities? We offer an example in [1].

¶ **idea/outline** There are two ideas to solve this question.

- **Parabolic rescaling**: Recall that $\lambda_i^2 g(t_i + \lambda^{-2} t)$ is preserved under RF, such as for $\lambda_i = |\mathrm{Rm}^{1/2}| \to \infty$ as $\quad t \to T$. Where we need
  - compactness theory for manifolds of Ricci flows.
  - desctiption of sing models (3D classification).
- **Surgery**

¶ **Cheeger-Gromov compactness**

definition 3.1. Let $(M, g_i, p_i) \to (M_\infty, g_\infty, p_\infty)$ a family smooth complete pointed Riemannian manifolds, if we have

$$p_\infty \in \mathrm{Int}(\Omega_1) \subset \Omega_1 \subset \Omega_2 \subset \cdots \subset M_\infty$$

a conpact exhaustion. Diffeomorphism onto their image

$$\phi : \Omega_i \to M_i \quad s.t. \phi(p_\infty) = p_i$$

and

$$\phi_i^* g_i \to g_\infty$$

is locally smoothly.

example 3.1. Asymptotically Euclidean: $(M^n, g_i, p_i) = (M^n, g, p_i) \to (\mathbb{R}^n, \delta_{ij})$.

remark 3.1. *If $(M_i, g_i, p_i) \to (M_\infty, g_\infty, p\infty)$ by CG, then*

$$\sup_{i \in \mathbb{N}} \sup |\nabla^k \mathrm{Rm}(g_i)| < \infty \quad \textit{for all } s > 0, k$$
$$\inf_i \mathrm{inj}(M, g_i, p_i) > 0$$

theorem 3.1 (Cheeger-Gromov). *Converse of the above is true up to talcing subsequences.*

¶ **compactness of Ricci flow**

theorem 3.2. *If $(M^n, g_i(t), p_i)_{t\in(a,b)}$ a sequence of RF s.t.*

- $$\sup_i \sup_{M\times(a,b)} |\mathrm{Rm}(g_i)| < \infty$$
- $$\inf_i \mathrm{inj}(M_i, g_i(0), p_i) > 0$$

then exists $(M^n_\infty, g_\infty, p_\infty)_{t\in(a,b)}$ a RF, s.t.

$$(M_i^n, g_i(t), p_i) \to (M^n_\infty, g_\infty(t), p-\infty)$$

in the sense:

- *compact exhaustion*
$$p_\infty \subset \mathrm{Int}(\Omega_1) \subset \Omega_2 \subset \cdots \subset M_\infty$$
- *$\phi : \Omega_i \to M_i$ a diffeomorphism onto image s.t. $\phi_i(p_\infty) = p_i$ and $\phi_i^* g(t) \to g(t)$.*

¶ **injectivity radius estimate** Recall, for singularity of compact RF, $t \to T$, choose $(x_i, \tilde{t}_i)$ s.t. $\sup_{M\times[0,t_i]} |\mathrm{Rm}| = |\mathrm{Rm}(x, \tilde{t}_i)|$. Consider rescaled RF

$$(M, Q_i g(t_i + tQ_i^{-1}), x_i)$$

and note that $|Rm_{g_i}| \le g$ (a uniform g).

† **Question** By rescaling can control $|Rm_{g_i}|$, but also need inj_{g_i} to control *volume noncollapsing*, i.e.

$$\mathrm{vol}(B(x,r)) \ge \alpha > 0$$

we introduce the *Poincare entropy functional*.

definition 3.2. Given a M, define a founctional

$$W(g,u,\tau) := \int_M \tau(4|\nabla u|^2 + Ru) - u^2 \log u^2 \mathrm{d}V_g - \frac{n}{2}\log\tau - \frac{n}{2}\log 4\pi - n$$

where $\int u^2 = 1$. And the *entropy* is defined by

$$\mu(g,\tau) := \inf_u W(g,u,\tau)$$

Then $\mu(g(t), T-t)$ is nondecreasing in t when $g(t)$ is a RF. Compute $\frac{d}{dt}W(g(t), u(t), T-t) \ge 0$ for $u(t)$ s.t. a hear equation and gives:

theorem 3.3 (no local collapsing)**.** *If $g(t)$ a RF on M(compact), $t \in [0,T]$, $p \in M$, r is sufficiently small s.t. $|R| \le r^{-2}$ on $B_t(p,r)$ then*

$$\frac{\mathrm{vol}(B_t(p,r))}{r^n} > \xi$$

¶ **singularity & blow up models**

theorem 3.4. *If $(M, g(t))_{t\in[0,T)}$ a maximal RF with $T < \infty$ and the $g_i(t)$ are defined as before, then*

$$(M, g_i(t), p_i) \to (M_\infty, g_\infty(t), p_\infty), \quad t \in (-\infty, b), \textit{ for some } b > 0$$

as RF and $|\mathrm{Rm}(g_\infty)|(p_\infty) = 1$, $|\mathrm{Rm}(\mathrm{g}_\infty)| \le 1$ *for* $t \le 0$.

† **Question** What singularity models occur?

In dimension $m = 2$, only S^2 or S^2/Γ in general, singularity models are ancient solutions. Hamilton's conjecture is that: "Most" singularities modelled on **shrinking** solitons.Let (M, g, X) be a shrinking soliton if $-2\mathrm{Ric} = -g + L_X g$, where $g(t) = |t|\phi_t^* g, \frac{d}{dt}\phi_t = X, \phi_0 = id, t < 0$.

In dimension $m = 3$, all shrinking solitons are S^3, $S^2 \times \mathbb{R}^2$, One can see the work of Perelmen in 2005 and Brendle in 2008. However, we hint that Bryant soliton is steady but not shrinking, for $-2\mathrm{Ric} = L_X g$.

Here we divide the singularities to **TWO** categories, in the case of $\sup_{t\to T} |\mathrm{Rm}|(T - t) > 0$.

- **Type-I**
$$\sup_{t\to T} |\mathrm{Rm}|(T - t) < \infty$$
- **Type-II**
$$\sup_{t\to T} |\mathrm{Rm}|(T - t) = \infty$$

For type-I, we have

- [24] and [16] have shown that type-I singularities of compact RF modelled by gradient shrinking solitons.
- [4] shows that "F-limits" of RF blow ups,
  - a smooth RF spacetime away from
  - a codim = 4 singularities set, singularities points of limit have blow ups are
    - Ricci flat cones;
    - gradient shrinkers.

3.2 reasons for Noncompact RF

¶ **understand & classify solitons & singularities models**

definition 3.3. A complete ancint RF $(M^n, g(t))_{t\in(-\infty,0]}$ s.t.

- $|Rm| \le 1$;
- $K - noncollapsing$ ($|Rm| \le r^{-1}$ on $B_r(x,t)$ imply $\mathrm{vol}(B_r(x,t)) \ge K \ge r^n$)

- $n = 2, 3$, $|Rm| \geq 0$, *Hamilton's Harnack inequality.*

are $K - solitons$.

remark 3.2. *We have following hints:*

- *classifying $K-$ solitons necessary for dimension of* 3 *RF with surgery. One can see [27], for no nontrivial compact shrinkers in* $n = 3$.
- *classify singularities models in dimension of* n.
- *classigy ancient solutions(or shrinking, steady, expanding).*

¶ examples of compact singularities When $n = 4$, we offer some examples.

example 3.2. [2] show that noncompact RF with singularities models are $Eguchi - Hanson$(ALE), $\mathbb{R}^4/\mathbb{Z}_2$, (Bryant/$\mathbb{Z}_2$ or $\mathbb{R}P^2 \times \mathbb{R}$).

example 3.3. [29] show that any asymptotically canonical noncompact gradient shrinking soliton appears as a singurities model of compact RF.

definition 3.4. (M^n, g) is asymptotically canonical to $(\mathbb{R}_t \times N^{n-1}, \mathrm{d}s^2 + s^2 h)$, if exists diffeomorphism $\Phi : (s, \infty) \times N \to M$ s.t.

$$|\nabla_s^k (\Phi_g^* - s)|_s \to 0 \quad \text{as } r \to \infty$$

remark 3.3. *Some rmks:*

- *more generally, compact* $(M^4, g(t))$ *encounters finite time singularities at* $T < \infty$, *then* $(T-t)^{-1} g(t)$ *converges to smooth compact gradient shrinker or some blow up converges to*
  - $S^2 \times \mathbb{R}^2$;
  - $(S^3/\Gamma) \times \mathbb{R}$;
  - *smooth Riemannian* ***Cone*** $g_\infty = ds^2 + s^2 h, \quad R_{g_\infty} \geq 0$.
- *For* $n = 4$, *the RF hopefully to apply to* $\pi_1(M) = 0$ *manifolds or* 4 *dimensional manifolds admitting* PSC.

¶ other applications

- Smooth noncompact metrics M asymptotically AE, $R \geq 0$, one can see in [22].
- Pinching of noncompact cases, where we introduce a conjecture of Hamilton as above.
- About AE.

theorem 3.5 (Hamilton's pinching conjecture)**.** *(M^3, g_0) is complete connected with* $\mathrm{Ric} \geq \epsilon R \geq 0$ *for some* $\epsilon > 0$, *then* (M^3, g_0) *flat or compact.*

One can see [9, 23, 21].

theorem 3.6. *If* $|\mathrm{Rm}| < \epsilon R$, (M, g) *is an AE, then RF flows to* $(\mathbb{R}^n, g^0)$.

One can see [10].

3.3 some Tools for Noncompact RF

¶ localized quantities & estimates

† **local entropy** One can see [31].

theorem 3.7. *If* $(M^n, g(t))_{t\in[0,T]}$ *is a RF s.t.*

$$\mathrm{Ric} \le \frac{(n-1)A}{n}, \quad x \in B_{g(t)}(x, \sqrt{t}),\ t \in (0,T]$$

then

$$\mu(\Omega_T, g(T), \tau) - \mu(\Omega_0, g(0)), \tau + T \ge -A^{-2}$$

for $\tau \in (0, A^2 T)$, *A is large, where* $\Omega_T = B_{g(T)}(x_0, \delta A\sqrt{T}), \Omega_0 = B_{g(0)}(x_0, 2\delta A\sqrt{T})$.

† **Nash entropy** One can see [19, 3]. Let M, and $\tau > 0$,

$$\mathrm{d}\gamma = (4\pi\tau)^{-n/2} d^{-f} \mathrm{d}V_g$$

and

$$N[g, f, \tau] = \int_M f \mathrm{d}\gamma - \frac{n}{2}$$

the pointed *Nash entropy* at (x_0, t_0) is

$$N_{x_0,t_0} = N[g_{t_0-\tau}, f_{t_0-\tau}, \tau]$$

where f s.t.

$$\mathrm{d}\gamma = (4\pi\tau)^{-n/2} e^{-f} \mathrm{d}g$$

is conjugate heat kernel

$$-\partial_t - \triangle + R = 0$$

and

$$\frac{d}{d\tau}(\tau N_{x_0,t_0}(\tau)) = W[g_0 - \tau, f_{t_0-\tau,\tau}] \le 0$$

† **pseudocality**

theorem 3.8. *Exists* $\epsilon, \delta > 0$ *s.t. if* $(M^n, g(t)), t \in [0, \epsilon r_0]$ *is a RF and*

- $R \ge -r_0^2$;
- $|\partial\Omega|^n \ge (1-\delta)c_n|\Omega|^{n-1}$ *for all* $\Omega \subset B_0(x_0, r_0)$ *open,* c_n *isop constant in* $\mathbb{R}^n$. *Then for any* $t \in [0, (\epsilon r_0)^2]$ *and* $x \in B_t(x_0, \epsilon r_0)$,

$$|\mathrm{Rm}|(x,t) \le t^{-1} + (\epsilon r_0)^{-2}$$

Which control $|\mathrm{Rm}|$ with local Euclidean property and R lower bounded.

¶ maximaum principle

theorem 3.9. *If $(M, g(t))$ is a complete solution to RF on noncompact one, with $|\mathrm{Rm}| \le K$ on $[0,T]$,*

$$\frac{\partial u}{\partial t} \le \triangle u + < \nabla u, X > + F(u,t), \quad |u(x,t)| \le \exp(A(d(0,X)+1))$$

and $U(t)$ solving

$$\frac{\mathrm{d}U}{dt} = F(u,t)$$
$$u(x,0) \le U(0) \quad \textit{for all } x \in M$$

then $u(x,t) \le U(t)$ for all $x \in M$, $t \in [0,T]$.

¶ weighted spaces ...

3.4 Sketch of some recent developments

¶ canonical expanders in 4 dimension Recall: compact 4 dimension RF with finite time singularities, blow up: compact smooth gradient strinker cylindrical, canonical cone γ with $R_\gamma \ge 0$.

† **Question** How to resolve 4 dimension canonical singularities? One can see [14, 17].

† **Question** Finding nonsymmetric cones?

$$\mathrm{Ric} + \frac{1}{2}L_X g + \frac{g}{2} = 0$$

is weakly elliptic, need to gauge.

$$Q(g) = \mathrm{Ric}_g + \frac{1}{2}L_X g + \frac{g}{2} - \frac{1}{2}L_{\mathrm{div}_g} - \frac{1}{2}\nabla \mathrm{tr} g$$

is strongly elliptic...

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