Mean curvature flow: An Introduction

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Preface

purpose: The purpose of this note is to record the mini lecture of mean curvature flow so that I can go back to review in the future, and also share some viewpoints of geometry flow to peers.

When I was a sophomore, participated in the 2024's SUMMER SCHOOL IN GEOMETRY in USTC and firstly learned **mean curvature flow** in the mini lecture called "Topics in mean curvature flow" teached by Dr.Zhu. I edited this lecture note which combined the blackboard and my own understanding during summer school time, thus it may have some errors.

I divide this note into **TWO** parts totally.

- The **FIRST** part is about introducing the mean curvature flow and some typical tools.
- The **SECOND** part mainly focus on singularity analysis and an special case: mean convex flow.

I consider a classical situation to introduce mean curvature and link it to an important object in category of PDE, and one may find much of common points between such flow and **heat equation**, that's nearly whole discussion in chapter1.

In chapter2, I mainly introduce a essential tool or property called **monotonicity formula**, which not only implies plenty of geometry and we will compare it with the problem of **area ratio** in category **minimal surface** but also "hide" a new operator called **conjugate heat operator** one can link it to the traditional one in PDE.

In the middle of note, I will show some of my viewpoints about singularity analysis, we can consider different geometry flow and sum up the method to research singularity. But I'm new to singularity analysis☹.

In chapter4, I will introduce some basic topics in **Geometry Measure Theory**, to prepare some of measure tools for the last chapter, one can learn more in [8].

In the end, I want to discuss an important flow in category of mean curvature flow, called **mean convex flow**, and show some great milestone-style works by Colding – Mincozzi.

Rmk:

1. To simplify the discussion, We just consider embedding type hypersurface in this note, If some of consequence can be extended to general cases, I will point out.
2. We ignore some of details in proof or whole proof.

3. This note will be updated in the future, you can download newest one in my homepage

$$https://zhenye-math.github.io/$$

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Chapter 1

An Introduction to Mean curvature flow

1.1 Motivation and Notations

1.1.1 A heuristic idea

¶ a simple relation between analysis and geometry First we consider a situation, one can regard it to the motivation of *minimal surface*, see [3]. Let Ω a region in $\mathbb{R}^n$ and a smooth mapping $u : \Omega \to \mathbb{R}^{n+1}$ without any singularities, so we can call this mapping a *hypersurface*[1]. Then one may focus on some metric consequences of it, like the length of curves on it or angle of them or area of the part of surface. But our question is that when the area get **maximal**.

We just write down the *Area variation* formula

$$\text{Area}(u) = \int_\Omega \sqrt{1 + |Du|^2}$$

where the operator D^2 means *derivative* in $\mathbb{R}^{n+1}$. One can actually get a excellent **extremum** of this variation, see [3], but we just focus on the approximate formula of it

$$\sqrt{1 + |Du|^2} \approx 1 + \frac{1}{2}|Du|^2, \quad \text{as the gradient } Du \text{ very small}$$

One can also regard this situation to a very local area of surface because of the smoothness. Then we can easily compute the *Euler − Lagrange equation* of it

$$\triangle u = 0$$

which is a typical *Laplacian equation*, see [9, 5, 2].

remark 1.1.1. *One can say that in a very local area, the minimal surface satisfies a Laplacian equation, vice versa, and we just need this "fuzzy" viewpoint.*

[1] If you have know about theory of *manifold*, it's no hard to add some topological conditions.

[2] In the following discussion, we let ∇ be *Levi − Civota connection* in the given Riemannian manifold, and there is no confusion with the *gradient* ∇.

After the situation of Analysis, one also can calculate condition of surface and get a geometry consequence, which means that **mean curvature** vanishing

$$H = 0$$

We ignore the process, one can see it in [3, 11].

Here we establish a simple relation between Analysis and Geometry, that is locally the Laplacian equation is "equal" to the minimal surface

$$\triangle u = 0 \Longleftrightarrow H = 0$$

¶ **more ideas** Then we consider situation further. In the theory of PDE, we have a kind of equation which links to deep physical implications, like the *heat conduction*, see [9, 2], i.e.

$$\triangle u = \partial_t u$$

Although we have plenty of method to research this equation, a natural question from the discussion above is that,

† **Question** Do we have some of **Geometry** relation to it?

One can be inspired by the introduction above, and just regard $\triangle u$ to H. Then it's natural to add $\partial_t u$ to the geometry one

$$H = \partial_t$$

If you have learned some of *Ricci flow* or other geometry flow theories, it's easy to establish a new idea of a flow about **mean curvature**, and that is the most significant topic of this note.

Rmk Actually, we did some tricks in former discussion. In fact, the development of PDE is earlier than mean curvature flow ☻, which is established by Brakke in 1978(see [1]). A well-known case is that we need lots of theories of PDE to solve the problem in geometry.

1.1.2 General cases

Now we extend these ideas to general cases. Firstly, let a embedding $M^n \to \mathbb{R}^{n+1}$ from any smooth $n-dimension$ manifold M to *Euclidean Space* of dimension $n+1$, and we call this embedding a $hypersurface$.

remark 1.1.2. *The reason that why we choose a embedding is we can find a* ***global*** *defined normal vector, because we need to consider the mean curvature. A extra advantage is that manifold can be oriented.*

Observed that M divides $\mathbb{R}^{n+1}$ into two part, that is M becoming a boundary of a open subset K in $\mathbb{R}^{n+1}$, i.e. $M = \partial K$. If M is compact, we can choose a *bounded* one to be K, see [11].

¶ **notations** Now we establish some notations.

- **second fundamental form(S.F.F)**[3]

$$A(X,Y) :=< D_X Y, \nu >\in \mathbb{R}, \quad X,Y \in \Gamma(TM),\ \nu \text{ is a global defined normal vector}$$

- **mean curvature** One can regard H to the *trace* of A, just choose a family of orthogonal unit basis $\{e_1, \cdots, e_n\}$, then[4]

$$H := \sum_{i=1}^{n} A(e_i, e_i) \in \mathbb{R}$$

Also a mean curvature *vector*

$$\overrightarrow{H} := H\nu$$

1.2 Mean curvature flow and some Examples

Now, it's natural to define the flow.

¶ **definition**

definition 1.2.1. Consider a family of embeddings $F : M^n \times I \to \mathbb{R}^{n+1}$ by $(x,t) \to F(x,t)$ where $x \in M, t \in I$[5], and $M_t := F(M,t)$ the image of embeddings. We call $\{M_t\}_{t\in I}$ a *mean curvature flow*(MCF) if

$$\partial_t F = \overrightarrow{H} = H\nu$$

definition 1.2.2. $M := \bigcup_{t\in I} M_t \times \{t\}$ is called $space-time\ track$ of mean curvature flow.

remark 1.2.1. *In fact, MCF can defined for **immersion** and **higher codimension** $F : M^n \to \mathbb{R}^{n+k}$, and*[6]

$$\partial_t F = \overrightarrow{H} = \sum_i (D_{e_i} e_i)^{\perp}$$

¶ **examples** There are **NOT** too many examples of MCF, but some of typical ones, one can see [1, 3, 4].

example 1.2.1 (Euclidean Space)**.** $\mathbb{R}^n$ just have *static* flow.

example 1.2.2 (Sphere)**.** Let $S^n(\sqrt{R^2 - 2nt})$ a $n-sphere$ in $\mathbb{R}^{n+1}$, with radius $r = R^2 - 2nt$. We need $R > 0$ and $t < R^2/2n$. It's easy to check that it is a MCF by the equation

$$\partial_t F = H$$

[3] In fact, we can define a general S.F.F on any embedding manifolds, but we just consider hypersurface of *codim* = 1. In this way, the S.F.F can not only be regarded as a scalar but also a vector, and we claim that the former one is independent of the direction of normal vector ν. The case of higher codimension is more complicate.

[4] remember that we don't average it by trA/n.

[5] $I = (a,b)$ or$[0,T)$ or$(-\infty,0)$

[6] Here we denote the projection to *normal bundle* by $\perp$ and the tangent one be $\top$.

where F is the position function of S^n. We find that the sphere *shrinks* to a point when $t \to R^2/2n$, which may be a difference compared with Ricci flow one, see [10].Observed that the derivative respect to t of position function F can be express by r, we have

$$|\partial_t F| = r' = \frac{n}{\sqrt{R^2 - 2nt}}$$

and obviously

$$H = \frac{n}{r} = \frac{n}{\sqrt{R^2 - 2nt}}$$

But one also can find a unique form of MCF in all radius S^n, just let $S^n(r(t))$, where $r(t)$ is the radius function just respect to time t. Because

$$|\partial_t F| = r', \quad H = \frac{n}{r}$$

so we have a ODE $rr' = n$.

remark 1.2.2. *If we use the space-time track on sphere model, just draw the S^1(remember general dimension) over the ancient one one by one. Finally we get a picture of "space-time", and the track is **parabolic**, because we can solve the t from*

$$r^2 = R^2 - 2nt$$

example 1.2.3 (Cylinder)**.** We define a *cylinder* by $S^k(\sqrt{R^2 - 2kt}) \times \mathbb{R}^{n-k}$, observed that the *principal curvature*s are

$$\{\underbrace{\frac{1}{r}, \cdots, \frac{1}{r}}_{k}, 0, \cdots, 0\}$$

so

$$H = \frac{k}{r} = \frac{k}{\sqrt{R^2 - 2kt}}$$

so does $|\partial_t F|$. One can find that if k(the "scale" of sphere) lower, the time shrinking to a point longer.

example 1.2.4 (Graph of function)**.** We define the *Graphcal* MCF[7]

$$\{(x(x,t), u(x,t)), x \in \mathbb{R}^n\}$$

and it's easy to compute

$$\nu = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}}$$

so does

$$-H = \operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right)$$

[7] Here the first x means a position vector in $\mathbb{R}^n$, and second one is also position but be regarded as an independent variable. when we do derivative of the "final" position vector F, must be careful that

$$\partial_t F = (\partial_t x, Du \cdot \partial_t x + \partial_t u)$$

And the derivative of u can be written from

$$\partial_t F \cdot \nu = -H$$

then

$$\partial_t u = \sqrt{1+|Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)$$

remark 1.2.3. *Via some computation, we can get*[8]

$$\partial_t u = \triangle u - \frac{D^2u(Du,Du)}{1+|Du|^2} \tag{1.1}$$

in the example above. It's easy to find that the second term in the right of equation tends to 0 *when Du is small, which verify our claim in section1.1 again!*

1.3 Avoidance principle

Here we introduce a very intuitional and accessible property of MCF. We claim that the MCF preserves the **non-touching** relation of two surfaces.

theorem 1.3.1 (Avoidance Principle)**.** *Let $\{M_t\}_{t\in[0,T)}$ and $\{N_t\}_{t\in[0,T)}$ be two MCFs of closed(at least one) hypersurfaces. If $M_0 \cap N_0 = \varnothing$, then for all $t \in [0,T)$, we always have $M_t \cap N_t = \varnothing$.*

In some books, it's called *Comparison Principle*, see [4].

Proof. If $M_t \cap N_t \neq \varnothing$ for the first time $t > 0$, then they must touch **tangentally** If not, they touched in the ancient time, since smoothness. in a small ball $B(\epsilon)$. Let $M = \mathrm{gragh}u$ and $N = \mathrm{graph}v$, then u, v both satisfy the formula1.1 in $B(\epsilon)\times[\epsilon^2,\epsilon^2]$. Then we let

$$\partial_t w = u - v = \triangle w + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\frac{D^2u^s(Du^2Du^s)}{1+|Du^s|^2}\mathrm{d}s$$

where

$$u^s = su + (1-s)v$$

which implies $\frac{\mathrm{d}}{\mathrm{d}s}u^s = u - v = w$. After computation we have

$$\partial_t w = \left(\delta_{ij} + \int_0^1 \frac{D_iu^sD_ju^s}{1+|Du^s|^2}\mathrm{d}s\right)D_{ij}w + \text{term}$$

In a small ball above, we claim the equation is uniformly elliptic, i.e.

$$a_{ij}D_{ij}w + \text{terms} \quad \text{s.t. } a_{ij}\xi^i\xi^j \geq \eta|\xi|^2, \text{ for } \eta > 0$$

Then we link it to the theory of parabolic.

[8] One can also get

$$\partial_t u = \left(\delta_{ij} + \frac{D_iuD_ju}{1+|Du|^2}\right)D_{ij}u$$

Recall the *Harnack inequality*, see[2, 9]. In a small ball, such as $B(\epsilon)$ above, we have

$$\sup w \le C \inf w, \quad \text{for some constant } C$$

Also $u \ge v$ implies $w \ge 0$ in $B(\epsilon)$, then $w = 0$ in a smaller ball. Finally, by the *strongly maximum principle* of parabolic equation, see [2, 9], we have

$$w \equiv 0$$

which implies conflict! □

Via the same method of proof above, we have following applications.

corollary 1.3.1. *The distance between two M in $\mathrm{Im}(F)$ is increasing.*

proof(sketch). By the same method of theorem, just let 0(touching) be having a distance $d > 0$. □

corollary 1.3.2. *The "embedding" is preserved under* MCF.

This corollary make sure that the MCF is well defined.

1.4 Short time existence

First we show a **FACT** that

$$\vec{H} = \triangle_{M_t} F$$

where F is the position function. Just observed that

$$\begin{aligned} 0 = D^2F(e_1, e_2) &= D_{e_i} D_{e_i} F - D_{D_{e_i} e_i} F \\ &= D_{e_i} D_{e_i} F - D_{\nabla_{e_i} e_i + A(e_i, e_i)\nu} F \\ &= D_{e_i} D_{e_i} F - D_{\nabla_{e_i} e_i} - < Df, \underbrace{A(e_i, e_i)\nu}_{\vec{H}} > \\ &= \triangle_{M_t} F - \vec{H} \end{aligned}$$

Thus by the MCF, we have

$$\partial_t F = \triangle_{M_t} F$$

One can discover that this is a "heat equation" but be careful to the Laplacian operator $\triangle_{M_t}$ is defined by the manifold M and changes respect to the time t. So that's **NOT** a traditional theory of *parabolic*. But we have following theorem.

theorem 1.4.1. *If M a closed hypersurface, then exists a unique* MCF $\{M_t\}_{t \in [0,\epsilon)}$ *s.t.* $M_0 = M$*, for some small* $\epsilon > 0$*.*

proposition 1.4.1. *The flow can be solved smoothly on* $(0, T)$[9] *but can **NOT** be solved on any* $(0, T')$*, for* $T' > T$*, i.e. exists a **maximal** existence time.*

[9]Here we ignore the start time, since the lack of smoothness.

proposition 1.4.2. *T is given, then we have*

$$\sup_{M_t} |A| \to \infty, \quad \text{as } t \to T$$

where $|A|^2 = \sum_{i,j=1}^n A(e_i, e_j)^2$.

Obviously that $|A|^2 \geq H^2/n$.

remark 1.4.1. *Be careful that here H may not tend to ∞! In fact, when $n = 2$, we have $H \to \infty$, but $n = 7$ that H is finite. Also the case of $2 < n < 7$ is an open question.*

Proof. If $|A| < C$ on $(0, T)$, then we have

$$|\nabla^l A| < C_{l,T}$$

on $(T/2, T)$ where $C_{l,T}$ is a constant respect to l, T. Where implies that M_t tends to M_T smoothly. Thus we have a MCF $\{M_t\}_{t \in [T, T+\epsilon)}$ starting from M_T. □

proposition 1.4.3. *the maximal existence time $T < \infty$.*

Proof. Just find a large ball to cover our surface M, it's no hard to have

$$M_t \subset B(\sqrt{R^2 - 2nt})$$

But we know that the ball must shrinks to a point when $t \to R^2/2n$, which means M_t is not smooth at $T = R^2/2n$. □

Chapter 2

Some simple Tools for MCF

2.1 Monotonicity formula

2.1.1 the formula

Here we introduce an important tool to research MCF, which is a monotonicity formula, and we can find the geometry intuition at the critical position.

remark 2.1.1. *Recall the classical differential geometry(see [11, 4]), we have known that given a **catenoid** M, then imagine a small ball touches it. In a very local case, we can guess the **Area Ratio***

$$\frac{\mathrm{vol}(B(\epsilon)\cap M)}{4\pi\epsilon^2} \to 1, \quad \text{as } \epsilon \to 0$$

But we also have

$$\frac{\mathrm{vol}(B(R)\cap M)}{4\pi R^2} \to 2, \quad \text{as } R \to \infty$$

because we can imagine the catenoid to be double planes in ∞, like a Wormhole.

theorem 2.1.1 (Monotonicity Formula, Huisken)**.**

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M \rho_{(x_0,t_0)}(x,t)\mathrm{d}\mu_t = -\int_M \left|\overrightarrow{H} + \frac{(x-x_0)^\perp}{2(t_0-t)}\right|^2$$

where

$$\rho_{(x_0,t_0)}(x,t) = \frac{1}{(4\pi(t_0-t))^{n/2}} e^{-\frac{|x-x_0|^2}{4(t_0-t)}}$$

*is called **Gaussian density**. Then we have*

$$\rho_{(x_0,t_0)}(x,t) \leq 0$$

so

$$\int_M \rho_{(x_0,t_0)}(x,t)\mathrm{d}\mu_t$$

*is **non-increasing**.*

2.1.2 proof and some ideas

¶ the proof of formula

Proof. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t}\rho \mathrm{d}\mu_t = \int_{M_t}\left(\partial_t\rho + \rho\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}\mu_t) + \triangle\rho + < D\rho, \overrightarrow{H} >\right)\mathrm{d}\mu_t \tag{2.1}$$

We divide the proof into three steps.

STEP1: compute the derivative of measure $\mathrm{d}\mu_t$: Observed that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}\mu_t) &= \frac{\mathrm{d}}{\mathrm{d}t}(\sqrt{g}dx^1\cdots dx^n)\\ &= \frac{1}{2}\sqrt{g}g^{ij}(\partial_t g_{ij})dx^1\cdots dx^n\\ &= \frac{1}{2}g^{ij}(\partial_t < \partial_i F, \partial_j F >)\mathrm{d}\mu_t\end{aligned}$$

where $g = \det(g_{ij})$. Since

$$\begin{aligned}\partial_t < \partial_i F, \partial_j F > &= 2 < \partial_i\partial_t F, \partial_j F >\\ &= 2 < \partial_i \overrightarrow{H}, \partial_j F >\\ &= 2\partial_i < \overrightarrow{H}, \partial_j F > -2 < \overrightarrow{H}, D_{ij}F >\\ &= -2 < \overrightarrow{H}, D_{\partial_i F}\partial_j F >\\ &= -2 < \overrightarrow{H}, \overrightarrow{A}(\partial_i F, \partial_j F) >\end{aligned}$$

Then by contraction of $\frac{1}{2}g^{ij}\partial_t g_{ij}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}\mu_t) = -|\overrightarrow{H}|^2$$

STEP2 compute the Laplacian of ρ $\triangle\rho$: Recall

$$\triangle_{M_t} f = \mathrm{div}_{M_t}(\nabla_{M_t} f)$$

We have

$$\triangle\rho = -\mathrm{div}\left(\rho\cdot\frac{(x-x_0)^\top}{2(t_0-t)}\right)$$

Then

$$\begin{aligned}\triangle\rho &= \frac{|(x_0-x)^\top|}{4(t_0-t)}\rho - \rho\mathrm{div}\left(\frac{(x-x_0)^\top}{2(t-t_0)}\right)\\ &= \frac{|(x_0-x)^\top|}{4(t_0-t)}\rho - \rho\left\{\mathrm{div}\left(\frac{(x-x_0)}{2(t-t_0)}\right) - \mathrm{div}\left(\frac{(x-x_0)^\perp}{2(t-t_0)}\right)\right\}\end{aligned}$$

Focus on that

$$\begin{aligned}
\operatorname{div}((x-x_0)^\perp) &= < D_{e_i} < x, x_0, \nu > \nu, e_i > \\
&= < x - x_0, \nu >< D_{e_i}\nu, e_i > \\
&= - < x - x_0, \nu > \underbrace{< \nu, D_{e_i} e_i >}_{\vec{H}} \\
&= -H < x - x_0, \nu >
\end{aligned}$$

then we have

$$\triangle \rho = \left\{ \frac{|(x-x_0)^\top|^2}{4(t_0-t)} - \frac{n}{2(t_0-t)} - \frac{H < x - x_0, \nu >}{2(t_0-t)} \right\} \rho$$

STEP3 compute other terms:

$$< D\rho, \vec{H} >= -\rho < \frac{x-x_0}{2(t_0-t)}, \vec{H} >$$

and

$$\partial_t \rho = \left(\frac{n}{2(t_0-t)} - \frac{|x-x_0|^2}{4(t_0-t)^2} \right) \rho$$

Finally we have

$$(\partial_t + \triangle - |\vec{H}|^2)\rho + < D\rho, \vec{H} >= - \left| \vec{H} + \frac{(x-x_0)^\perp}{2(t_0-t)} \right|^2 \rho$$

□

remark 2.1.2. *Recall the formula2.1 in the proof. In fact we only have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{M_t} \rho \mathrm{d}\mu_t = \int_{M_t} \left(\partial_t \rho + \rho \frac{\mathrm{d}}{\mathrm{d}t} (\mathrm{d}\mu_t) + < D\rho, \vec{H} > \right) \mathrm{d}\mu_t$$

which implies a operator

$$\partial_t - |\vec{H}|^2$$

but we can't get a square of $\left| \vec{H} + \frac{(x-x_0)^\perp}{2(t_0-t)} \right|$*! Observed that the* Laplacian *operator* $\triangle$ *vanishing in the integration.*

¶ **conjugate heat equation** One may find an interesting operator in our proof,

$$\partial_t + \triangle - |\vec{H}|^2$$

which is called *conjugate heat kernel*. Then we transform this viewpoint to analysis, and compare it with traditional heat equation. We define an operator

$$\square^* = \partial_t + \triangle + |\vec{H}|^2$$

where the heat kernel is

$$\square = \partial_t - \triangle$$

and the former one links to the *conjugate heat* equation. We show a proposition.

proposition 2.1.1. *If* $\square u = 0$ *and* $\square^* v = 0$*, then*

$$\int_{M_t} uv \mathrm{d}\mu_t = \text{const}$$

where M_t *is a closed* MCF.

¶ **geometry of the formula** Then we want to discover the geometry of theorem2.1.1. Observed that the integration of Gaussian density, if $\int \rho \mathrm{d}\mu_t \equiv \text{const}$, then

$$\overrightarrow{H} + \frac{(x - x_0)^\perp}{2(t_0 - t)} = 0$$

This relation is suitable for any codimension, but for $1 - codim$, we have a simple expression by inner product

$$H + \frac{< x - x_0, \nu >}{2(t_0 - t)}$$

To explain this phenomenon, we introduce a new concept.

definition 2.1.1. A MCF is a $self - shrinker$, if

$$M_t = \sqrt{-t} M_{-1}, \quad t < 0$$

example 2.1.1. A sphere above will be written by $\sqrt{-2nt}$.

proposition 2.1.2. MCF *is a self-shrinker if and only if*

$$H + \frac{< x - x_0, \nu >}{-2t} = 0, \quad \textit{for some } t < 0$$

Via this proposition, we claim that, if we let the formula to be "=", then we get a shrink!

Proof. Sufficiency. Let $\tilde{F}(x,t) = \sqrt{-t}\tilde{F}(x,-1)$, where $\tilde{F} : M \times (-\infty, 0) \to \mathbb{R}^{n+1}$. But the direction of $\partial_{\tilde{F}}$ is not to the normal one, so we need to modify it. Just consider

$$(\partial_t F)^\perp$$

and compute

$$\partial_t \tilde{F} = \frac{1}{2\sqrt{-t}} \tilde{F}(x,-1) = \frac{1}{2}\tilde{F}(x,t)$$

then

$$(\partial_t F)^\perp = \frac{1}{2}(\tilde{F})^\perp = \overrightarrow{H}$$

Necessity. If $H + \frac{<x-x_0,\nu>}{-2t} = 0$, just take $M_s = \sqrt{\frac{s}{t}} M_t$. □

Then we go back to $minimal\ surface$, we find that the area ratio

$$\frac{\mathrm{vol}(B(r)) \cap M}{4\pi r^2}$$

is increasing from 1 to 2. And we let the position x to λx for some $\lambda \neq 0$, the area ratio invariant, that is the area ratio is an invariance under the trivial rescaling, so that is a **cone**.Now we look at theorem2.1.1, it's easy to find that the shrink is a **parabolic cone**, that is for $space - time$ $(x,t) \to (\lambda x, \lambda^2 t)$.

2.2 Gaussian density

We introduce a new notation.

definition 2.2.1. We define a function respect to a fixed position x_0, fixed time t_0 and a free variable about periods of time r^2

$$\Theta_M(x_0,t_0,r) := \int_{M_{t_0-r^2}} \frac{1}{(4\pi)^{n/2}r^n} e^{-\frac{|x-x_0|^2}{4r^2}} \mathrm{d}\mu_{t_0-r^2}$$

to be *Gaussian density.*

One can compare this formula with the one in theorem2.1.1, just let $r^2 = t_0 - t$. It's easy to find that r is not only the *variance* of Θ If you have learned something about measure theory of probability but also a time scale between t_0 and t, and Θ is increasing as r is **increasing**.

If we want to describe a very local area of t_0, we can get

$$\Theta_M(x_0,t_0) := \lim_{r\to 0} \Theta_M(x_0,t_0,r)$$

which is independent of the scale of t.

remark 2.2.1. *We can get*

$$\Theta_{\mathbb{R}^n}(x_0,t_0,r) = \begin{cases} 1 & x_0 \in \mathbb{R}^n \\ 0 & x_0 \notin \mathbb{R}^n \end{cases}$$

One can describe this formula to be characteristic function, but the 1 *is not essential, that's why we choose a scalar* 4π *in the formula.*

Another more geometrical explanation is that the density formula Θ is a **area ratio**, which we have implied in section2.1. And we introduce a famous consequence of Brakke in 1978, one can see [1].

theorem 2.2.1 (Brakke regularity)**.** *Let* $M = \{M_t\}$ *be a* MCF*, for* $(x_0,t_0) \in M_{t_0}$*. If* $\Theta(x_0,t_0,r) < 1+\epsilon$*, then*

$$|A| \leq Cr^{-1}$$

in $B(x_0,\epsilon r) \times [t_0 - \epsilon^2 r^2, t_0 + \epsilon^2 r^2]$

Since the Θ is an invariance under rescaling, it's no hard to let $r = 1$. Intuitively, we say that if the area ratio is closed to 1, then the surface won't bend too much.

2.3 Colding-Mincozzi Entropy

Now we introduce the work of Colding and Minicozzi.

¶ Colding-Mincozzi entropy

definition 2.3.1. We define *Colding – Minicozzi entropy*

$$\mathrm{Ent}(M) := \sup_{x_0\in\mathbb{R}^{n+1}, r>0} \int_M \frac{1}{(4\pi)^{n/2} r^n} e^{-\frac{|x-x_0|^2}{4r^2}} \mathrm{d}\mu$$

without a period of time, but for all x, t.

proposition 2.3.1. *Let* $\{M_t\}$ *a* MCF*, then* $\mathrm{Ent}(M)$ *is not increasing.*

Proof. $\mathrm{Ent}(M_t)$ is achieved at (x_0, t_0), i.e.

$$\mathrm{Ent}(M_t) := \sup_{x_0\in\mathbb{R}^{n+1}, r>0} \int_{M_t} \frac{1}{(4\pi)^{n/2} r^n} e^{-\frac{|x-x_0|^2}{4r^2}} \mathrm{d}\mu$$

for all $s < t$, we have

$$\mathrm{Ent}(M_t) \le \int_{M_s} \frac{1}{(4\pi(r^2+t-s))^{n/2}} e^{-\frac{|x-x_0|^2}{4(r^2+t-s)}} \mathrm{d}\mu_s = \mathrm{Ent}(M_s)$$

□

remark 2.3.1. *But be careful that* entropy ≡ const *may not imply a shrink! One can compare it with the theory in section2.1.*

However, we have following proposition.

proposition 2.3.2. *Finite entropy if and only if* M_t *have* ***euclidean volume growth****, i.e.*

$$\frac{\mathrm{vol}(B(r)\cap M_t)}{r^n} \le C$$

where C *is independent of* r*.*

Proof. We just prove necessity. For all $r > 0$, $\mathrm{Ent}(M) < C$. Then

$$\begin{aligned} C \ge \mathrm{Ent}(M) &\ge \int_M \frac{1}{(4\pi r^2)^{n/2}} e^{-\frac{|x-x_0|^2}{4\pi r^2}} \\ &\ge \int_M \frac{1}{(4\pi r^2)^{n/2} e^{-1/4}} \\ &= \frac{e^{-1/4}}{(4\pi)^{n/2}} \frac{\mathrm{vol}(B(r)\cap M)}{r^n} \end{aligned}$$

□

¶ famous consequence

theorem 2.3.1 (Colding – Minicozzi)**.** *Only* ***polynomial*** *volume growth entropy stable shrinkers are*

$$S^k \times \mathbb{R}^{n-k}, \quad k = 0, 1, \cdots, n$$

This theorem links the entropy to the shrinks.

remark 2.3.2. *In fact, most of our examples are polynomial volume growth. And the stable means that any small perturbastion will increase entropy.*

proposition 2.3.3 (Colding – Minicozzi)**.** *For shrinker* $M_t = \sqrt{-t}M_{-1}$*, we have entropy*

$$\mathrm{Ent}(M_{-1}) = \int \frac{1}{(4\pi)^{n/2}} e^{-|x|^2/4}$$

i.e. entropy is achieved at $(x_0, r) = (0, 0)$*.*

Chapter 3

some viewpoints in Singularities Analysis

¶ singularity phenomenon Recall proposition1.4.2, we have

$$\sup_{M_t} |A| \to \infty, \quad \text{as } t \to T$$

Which means that we may have singularities near the maximal existence time T. This troublesome situation is also existing in other geometry flow such as Ricci flow, see [10].

One can consider two $n-sphere$ do *connected sum* with a *cylinder* $S^k \times \mathbb{R}^{n-k}$, and it's shaped like a dumbbell. We have known that these two models can equipped with MCF

$$R^2 - 2nt,\ r^2 - 2kt, \quad r << R$$

and the maximal time can be computed easily. Here the $collapse - time$ of cylinder is much lower than sphere. Intuitively, we may get a singularity at the link of two deforming spheres. A key step of research geometry flow is researching the singularities, our methods are $rescaling$, $surgery$ see [10] etc. Similarly in the category of MCF, we attempt a way called $parabolic\ rescaling$ to present the singularities.

¶ parabolic rescaling Give any MCF $\{M_t\}_{t\in I}$, we can perform a parabolic rescaling based at (x_0, t_0) by

$$M_t^\lambda := \underbrace{\lambda}_{magnify} (M_{t_0 + \underbrace{\lambda^{-2}t}_{slowdown}}, -x_0)$$

Here we must be careful that not only add a factor λ to $magnify$ the local area of singularity but also control the speed of flow! Which makes the new object M_t^λ be a flow. In this way, we define the function of it

$$F_t^\lambda(x,t) := \lambda(F(x, t + \lambda^{-2}t) - x_0)$$

proposition 3.0.1. *$\{M_t^\lambda\}$ is actually a* MCF.

Proof. Consider

$$\partial_t F^\lambda = \lambda \cdot \lambda^{-2} \partial_t F = \lambda^{-1} \overrightarrow{H}_{M_t} = \overrightarrow{H}_{M_t^\lambda}$$

□

proposition 3.0.2. *Self-shrinker is invariant along parabolic rescaling based at* $(0,0)$.

Proof. Consider

$$M_t^\lambda = \lambda M_{\lambda^{-2}t} = \lambda\sqrt{-\lambda^{-2}t}M_{-1} = \sqrt{-t}M_{-1} = M_t$$

□

We can also find that the shrink is a *cone* under the parabolic rescaling.

definition 3.0.1. the $space - time\ track$ of M_t^λ is defined by

$$D_\lambda M := \bigcup_{t\in\lambda^2 I} \lambda M_{\lambda^{-2}t} \times \{t\}$$

Via proposition3.0.2, we claim that if M is a space-time track of a shrink then

$$D_\lambda M = M$$

Recall the formula

$$\Theta_M(x_0, t_0, r) = \int_{M_{t_0-r^2}} \frac{1}{(4\pi r^2)^{n/2}} e^{-\frac{|x-x_0|^2}{4r^2}}$$

then we have

$$\Theta_M(0,0,r) = \Theta_{D_\lambda M}(0,0,r)$$

Chapter 4

Some topics in Geometry Measure Theory

4.1 Brakke flow

¶ the relation to Radon measure Let M a hypersurface, we can get a *Radon measure* μ by defined

$$\mu(A) := \int_{A\cap M} \mathrm{d}\mu$$

† **Question** How can we find a way to get a surface from a given measure?

Maybe we are short of *regularity*. So we need to consider a tangent plane of a given measure μ. Just define a

$$\mu_{x,\lambda}(A) := \frac{\mu(x+\lambda A)}{\lambda^n}$$

We say a Radon measure μ is $n-rectifiable$, if

$$\lim_{\lambda\to 0} \mu_{x,\lambda} = \theta H^n LP$$

where θ is multiple numbers, H is $Hausdorff\ measure$, and P is the tangent plane $T_x\mu$. L means that restricts to the tangent plane, that is

$$H^n LP(A) := H^n(A\cap P)$$

theorem 4.1.1. *If μ is integer $\theta\in\mathbb{Z}$ n-rectifiable if and only if*

$$\mathrm{supp}\mu = \textit{union of } C^1 \textit{ manifolds} \bigcup \textit{measure } 0 \textit{ set}$$

remark 4.1.1. *Because our family of surface may not limiting, but any measure does, so we just define a measure with some weak conditions to get a surface on the contrary.*

¶ Rrakke flow Then we can define the Rrakke flow.

definition 4.1.1. A family of Radon measure $\{\mu_t\}_{t\in I}$ is called $Brakke\ flow$, if μ_t is integer n-rectifiable **almost everywhere** and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f\mathrm{d}\mu_t \le \int -|\overrightarrow{H}|^2 + < Df, \overrightarrow{H} > \mathrm{d}\mu_t \tag{4.1}$$

where $\forall f \in C_0^\infty(\mathbb{R}^{n+1})$. And

$$\int \mathrm{div}_{T_x\mu}(Df)\mathrm{d}\mu_t = \int < Df, \overrightarrow{H} > \mathrm{d}\mu_t$$

for any $f \in C_0^\infty(\mathbb{R}^{n+1})$ and $\mathrm{div}(Df) =< D_{e^i}Df, e_i >$.

remark 4.1.2. *One may confuse that the H here with the mean curvature we discussed above, but latter one is* ***NOT*** *given and just satisfies the conditions.*

One may also confused that why we choose such terms and a inequality but not equation in the formula4.1. Actually, we can compare it with the monotonicity formula2.1. Recall

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t} \rho \mathrm{d}\mu_t = \int_{M_t}\left(\partial_t\rho + \rho\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}\mu_t) + \triangle\rho + < D\rho, \overrightarrow{H} >\right)\mathrm{d}\mu_t$$

we find the terms, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}\mu_t) = |\overrightarrow{H}|^2, \quad < D\rho, \overrightarrow{H} >$$

so we just choose two typical terms in monotonicity formula to "Axiomatically defined" a general mean curvature flow. Because we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f \mathrm{d}\mu_t \leq \int -|\overrightarrow{H}|^2 + < Df, \overrightarrow{H} > \mathrm{d}\mu_t$$

However, one can actually regard the Brakke flow to the mean curvature flow, for some regular cases like spheres. And the question about inequality is that we may not preserve the equation in some limitation.

remark 4.1.3. Brakke *flow can disapper abruptly.*

theorem 4.1.2 (Brakke Compactness theorem)**.** *If $\{\mu_t\}_{t\in I}$ is a sequence of integral Brakke flow, then passing to subsequences $\mu_t^i \to \mu_t$.*

This theorem guarantee that the category of Rrakke flow is "closed". One may imagine that

$$\text{MCF} \subset \text{Brakke flow} \subset \text{Radon measure}$$

namely the objects in the middle term **CANNOT** escapes from it, but objects in the left term can escape, that's one of reasons why we need to construct a new category called Brakke flow.

remark 4.1.4. *In fact, the* Brakke *compactness is* ***TRUE*** *if we consider unit regular cyclic* Brakke *flow. What is the meaning of it?*

Let $M = \{M_t\}_{t\in[0,T)}$, then

$$\sup_{M_t}|A| \to \infty, \quad \text{as } t \to T$$

As we have shown in chapter3, the *blow up* analysis at (p, T) where we always assume that p is a singularity takes $(x_t, t_i) \to (p, T)$ s.t.

$$A(x_i, t_i) \to \infty$$

Let $M_t^i = \lambda_i(M_{t_i+\lambda^{-2}t} - x_i)$, and M^i is the space-time track of it. Via theorem4.1.2, we can transform $M_t^i \to M_t^\infty$ as $i \to \infty$ to an *unit regular cyclic* Brakke flow.

4.2 Limit flow and Tangent flow

definition 4.2.1. $\{M_t^\infty\}_{t\in I}$ is called a *limit flow*. If $(x_i, t_i) \equiv (p, T)$, then any limit M_t^∞ is called *tangent flow*.

For example, if the local area of a surface is smooth, then the tangent flow will tend it to a "plane", but if the area is not too much smooth, then the flow tend it to a "cone". One can observe that $t_i + \lambda_i^{-2} t \geq 0$, which implies $t \in [\lambda_i^2 t_i, *)$ then

$$t \in (-\infty, *]$$

so such solution is a *ancient solution*. Thus the research of singularities can ref to research ancient solution, one can compare it with chapter3.

theorem 4.2.1 (Huisken)**.** *Tangent flow is self-shrinker.*

Proof. Let $(x_i, t_i) \equiv (0, 0)$, then

$$\Theta_{M^i}(0, 0, r) = \Theta_M(0, 0, \lambda_i^{-2} r)$$

observed that the left term links to Θ_{M^∞}, and we have

$$\lim_{s \to 0} *(0, 0, s) = \Theta_M(0, 0)$$

Via theorem2.1.1, we have "="

$$0 = \Theta = \frac{\mathrm{d}}{\mathrm{d}t} \int_{M_t} \rho \mathrm{d}\mu_t = \int -\left| H + \frac{x^\perp}{-2t} \right|^2$$

then it is a shrink. □

theorem 4.2.2 (Ilmianen)**.** *If $n = 2$, tangent flow is smooth.*

remark 4.2.1. *But it is not true for some higher dimension, such as $n = 7$, the one* ***CANNOT*** *be smooth, one can see* Simon*'s cone*

$$\{x_1^2 + \cdots x_4^2 = x_5^2 + \cdots + x_8^2\}$$

Chapter 5

Convexity Theory

5.1 Mean convex MCF

¶ **convex hypersurface** In this chapter, we mainly focus on the *convex* case of MCF. Recall, in the category of differential geometry, we have defined a linear operator on hypersurface, called *Weingarten operator* which is a self-adjoint one and the *spectrum* can be written

$$\mathrm{spec}W := \{\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n\}$$

where λ_i can be regarded as a function respect to the point of surface, but be careful that such functions are continuous but not C^1, expect for the cases without umbilical points, see [11]. We call λ_1 the *first principal curvature* of W.

definition 5.1.1. We say a hypersurface is *convex* if $\lambda_1 \ge 0$.

It is a basic concept in differential geometry, but we hint that if $\lambda \ge \delta > 0$, then the surface(closed) will cover a domain as we have discussed in section1.1. One can easily find that the bounded domain K is a **convex** set in the category of set theory, which means that if $x, y \in K$, then $tx + (1-t)y \in K$ for $t \in [0,1]$.. Now we start to discuss the convex type singularities.

theorem 5.1.1 (Ball theorem,Huisken)**.** *the* MCF *of convex hypersurface shrinks to a round point p, i.e. tangent flow at* (p,T) *is* $S^n(\sqrt{-2nt})$.

One can link this theorem to a classical consequence in global differential geometry, that is *Liebamnn theorem*, see [11]. Also, a more similar consequence is about constant mean curvature.

theorem 5.1.2 (Hopf)**.** *Let* M *a compact, connected and closed surface in* $\mathbb{R}^3$ *with constant mean curvature, if* $M \cong S^2$*, then* M *is actually a standard sphere.*

proposition 5.1.1. *If* M_0 *is convex, then* M_t *is convex* $t > 0$*, whenever* M_t *is defined.*

That is the convexity is preserved along MCF.

¶ **tricks** Here we let $A^\sharp$ a $(1,1)-$type tensor, s.t.

$$< A^\sharp(X), Y >:= A(X,Y)$$

where A is S.F.F, but sometimes we write $A^\sharp = A$.

proposition 5.1.2.

$$\partial_t A = \triangle A + |A|^2 A$$

Let C a set of all *symmetric* matrices with *eigenvalues* $\lambda \geq 0$. If $\lambda > 0$, then $\mu\lambda > 0$ for $\mu > 0$, so C is a cone.

definition 5.1.2. We say C is a *invariant cone*, if it is invariant under $O(n)$ action, namely if $A \in C$, then $PAP^T \in C$.

Which makes sure that we can change the basis freely in C. And in our cases, C is denoted by invariant, convex cone.

definition 5.1.3. We say S.F.F$A \in C$, if the matrix $< A(e_i), e_j > \in C$.

the invariant property makes sure the definition is well.

5.2 ODE-PDE Principle

Now we introduce an important tool in the analysis of flow. One can compare it with the one in Ricci flow, see [10].

theorem 5.2.1 (ODE-PDE Principle,Hamilton)**.** *If C is a invariant cone, and is invariant along the evolution of ODE*

$$\frac{\mathrm{d}}{\mathrm{d}t} A = |A|^2 A$$

then it is invariant along the evolution of PDE

$$\partial_t A = \triangle A + |A|^2 A$$

Recall proposition5.1.2, one can also compare it with the *maximum modulus principle*, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} A_{\min} \geq |A_{\min}|^2 A_{\min}$$

remark 5.2.1. *Invariant under ODE means that if $A(0) \in C$, then $A(t) \in C$. But for PDE case, if $A|_{t=0} \in C$, for all $x \in M_0$, then $A(x,t) \in C$, for all $x \in M_t, t > 0$.*

proof of proposition5.1.1. By theorem5.2.1, we just verify the case in ODE. Consider

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \begin{bmatrix} |A|^2\lambda_1 & & \\ & \ddots & \\ & & |A|^2\lambda_n \end{bmatrix}$$

Obviously preserved in ODE, so does PDE. $\square$

5.3 some Consequences

Now we extend the concept of *convex*.

definition 5.3.1. M is called $k-convex$, if $\lambda_1 + \cdots + \lambda_k \geq 0$.

Same method of proposition5.1.1, we have

theorem 5.3.1. *$k-convex$ is preserved under* MCF.

An immediate corollary is that the mean curvature $H \geq 0$ is preserved under MCF.

proof of theorem5.1.1. Recall an important estimate from Huisken

$$\frac{|A|^2 - \frac{1}{n}H^2}{H^{2-\delta}} \leq C$$

then

$$\left(\frac{|A|^2}{H^2} - \frac{1}{n}\right) H^\delta \leq C$$

let $A \to \infty$, we have $H \geq |A|$, then

$$\frac{|A|^2}{H^2} - \frac{1}{n} \to 0$$

tangent flow at singularity s.t.

$$\frac{|A|^2}{H^2} = \frac{1}{n}$$

Via

$$\sqrt{\lambda_1^2 + \cdots + \lambda_n^2} \geq \frac{H}{\sqrt{n}}$$

" = " iff $\lambda_1 = \cdots = \lambda_n$, that is a sphere. ☐

definition 5.3.2. MCF is *mean convex*, if $H \geq 0$.

proposition 5.3.1 (Jacobi equation)**.**

$$\partial_t H = \triangle H + |A|^2 H$$

remark 5.3.1. *Recall theorem5.2.1, one can find this formula just the trace of it.*

Proof. Let $M_t^s := M_{t+s}$, and link to $F^s(-,t)$. And

$$\partial_t M_t^s = f\nu$$

for some function on M. Compute

$$\partial_t \partial_s F^s = \partial_t(f\nu) + \underbrace{f(\partial_t \nu)}_{\perp \nu}$$

and

$$\partial_s \partial_t F^s = \partial_s(H\nu) = (\partial_s H)\nu + \underbrace{H(\partial_s \nu)}_{\perp \nu}$$

Via variation of mean curvature, we have

$$\partial_t f = \partial_s H = \triangle f + |A|^2 f$$

Finally, observed that f in general, so let $f = H$. ☐

In fact, one can let any function, such $f =< \overrightarrow{w}, \nu >$, then

$$\partial_t < \overrightarrow{w}, \nu >= \triangle < \overrightarrow{w}, \nu > + |A|^2 < \overrightarrow{w}, \nu >$$

remark 5.3.2. *Last but not least, we claim that $t > 0$, then must $H > 0$, for* ***elliptic point*** *on closed surfaces.*

In the end, we show a great milestone-style consequence, and one can compare it with theorem2.3.1.

theorem 5.3.2 (Huisken,Colding – Mincozzi)**.** *Mean convex shrinker is*

$$S^k \times \mathbb{R}^{n-k}, \quad \text{for } k = 0, 1, \cdots, n$$

The proof is intrinsic, we ignore most of calculate, but show the last step

$$0 = \lambda_i \lambda_j, \quad \text{if } \lambda_i \neq \lambda_j$$

then just two cases:

1. $\lambda_1 = \cdots = \lambda_n > 0$, then S^n
2. $\lambda_1 = \cdots = \lambda_{n-k} = 0$ and $\lambda_{n-k+1} = \cdots = \lambda_n > 0$, then $S^k \times \mathbb{R}^{n-k}$.

5.4 some topics of Mean Convex flow

5.4.1 Mean convex flow

It's no hard to let $H > 0$ in this setting.

¶ noncollapsing

definition 5.4.1. Given a smooth embedded mean convex hypersurface M^n, the *insribe radius* is defined by

$$r(x) := \sup\{B(\underbrace{x + r\nu}_{center}, \underbrace{r}_{radius}) \subset K\}$$

where K is an open domain bounded by M.

We can estimate the radius

$$r(x) \leq \frac{1}{\lambda_n(x)}$$

so the curvature of surface will also influence radius. We also define

$$\mu(x) := \frac{1}{r(x)} \geq \frac{1}{\lambda_n}$$

And we need a definition to avoid the case of collapsing, such having two *sheets*.

definition 5.4.2. The mean convex MCF $\{M_t\}_{t \in I}$ is $\alpha - noncollapsed$, for $\alpha > 0$, if

$$r(x, t) \geq \alpha H(x, t)^{-1}$$

theorem 5.4.1. *On a mean convex* MCF *of closed hypersurface*

$$\partial_t \mu \le \triangle \mu + |A|^2 \mu$$

in viscosity sense.

remark 5.4.1. *Indeed, by* Brendle,

$$\partial_t \mu \le \triangle \mu + |A|^2 \mu - \frac{(\nabla_i \mu)^2}{\mu - \lambda_i}$$

For principal direction $\{e_1, \cdots, e_n\}$, $(\nabla_i \mu := \nabla_{e_i} \mu)$.

corollary 5.4.1. *Mean convex* MCF *of closed hypersurface is* $\alpha-$*noncollapsed.*

5.4.2 Convex estimate

theorem 5.4.2 (convexing estimate,Huisken – Sinestrari)**.** *Given a* MCF *of mean convex closed hypersurface* $\{M_t\}_{t\in I}$, *then for all* $\eta > 0$, *exists* C_η *s.t.*

$$\lambda_1 \ge -\eta H - C_\eta$$

on M_t.

corollary 5.4.2. *In mean convex flow*

$$\liminf \frac{\lambda_1}{|A|} \ge 0, \quad \text{as } |A| \to \infty$$

remark 5.4.2. *The mean convex limit flow is defined on ancient solution as we have discussed above. Goal of this section is to prove an important structure theorem.*

We call the flow of sequence t_i before singularity time T is $special\ flow$.

theorem 5.4.3 (White's structure theorem)**.** *Any special limit flow* $\{M_t^\infty\}_{t\in(-\infty,T)}$ *of mean convex* MCF *is an ancient solution that is*

1. *smooth and convex until extinction at* T*;*
2. *noncollapsed;*
3. *blow down limit is a unique cylinder* $S^k(\sqrt{-2kt}) \times \mathbb{R}^{n-k}$.

The part of "smooth" is too much difficult to prove, we ignore. And the part of "noncollapsed" we have proved in corollary5.4.1.

remark 5.4.3. *Where the* ***blow down*** *is an inverse step to blow up.*

$$M_t^i = \lambda_i (M_{\lambda_i^{-1}} t - x_0), \quad \text{as} \lambda \to 0$$

So

$$\Theta_{M^\infty}(0,0,r) = \lim_{\lambda_i \to 0} \Theta_{M^i}(0,0,r) = \lim_{\lambda_i \to 0} \Theta_M(0,0,\lambda_i r)$$

Via this theorem, all limit flows are smooth, so does ancient solutions, see section4.2.

5.4.3 Ancient solutions

definition 5.4.3. The hypersurface M s.t. $H = < \overrightarrow{w}, \nu >$ is called *translator* $M_t = M + t\overrightarrow{w}$.

Recall

$$H + \frac{< x, \nu >}{2} = 0$$

links to shrinker.

example 5.4.1 (Graphical translator)**.** For $\{x \times u(x) | x \in \mathbb{R}^n\}$

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = \frac{1}{\sqrt{1 + |Du|^2}}$$

example 5.4.2. Let $u = u(|x|)$, $u'(0) = 0$, then exists rotationally symmetric translator, called *bowl soliton*

$$u(|x|) \approx x^2 + \text{l.o.t}$$

theorem 5.4.4 (Brendle – Choi)**.** *The only uniformly* $2-$*convex, noncompact noncollapsed ancient solution is bowl soliton or* $S^{n-1} \times \mathbb{R}$.

remark 5.4.4. *In* $\mathbb{R}^3$*,* ***NO*** *need to be* $2-$*convex.*

theorem 5.4.5 (Angenent – Dankalopoulos – Sesum)**.** *The only uniformly* $2-$*convex compact noncollapsed solution is* S^n *and ancient ovel.*

5.5 Level set flow

† **Hint1** The *level set flow* will tell you how to "traverse" singularity.

So here we note the method to research singularities, recall chapter3:

- rescaling;
- surgery;
- level set flow.

† **Hint2** We just let the setting to be *codim* = 1.

The step of use this method is find a **function** v satisfies the set

$$\{x | v(x, 0) = 0\}$$

is given hypersurface M_0 exactly. That's why we only consider *codim* = 1. That is satisfies a PDE

$$\partial_t v = |Dv| \cdot \operatorname{div}\left(\frac{Dv}{|Dv|}\right)$$

If v is smooth, then $M_t \longleftrightarrow \{v(-, t) = 0\}$, then M_t is MCF. In other words, we transform the problem of surface geometry flow into the problem of "function flow". $Dv = 0$, with singularities, I guess these ideas can be applied by *Sard – Brown* theory and *Morse* theory, see[6, 7].

¶ **arrival-time function(just for convexity)** If M_0 is mean convex, then

$$H = \operatorname{div}\left(\frac{Dv}{|Dv|}\right) = -\frac{1}{|Dv|} = \partial_t x$$

definition 5.5.1. Let

$$M_t =: \{x|v(x) = t\}$$

If M_t is smooth then M_t is a MCF, where $M_0 = \{x|v(x) = 0\}$ we call this M_t a *level set flow*.

One can imagine this flow just the continuous "flow" of **sections** of a "mountain", and it naturally pass through the "singularity", but may not be smooth.

Bibliography

[1] Kenneth A Brakke. *The motion of a surface by its mean curvature.(MN-20)*, volume 20. Princeton University Press, 2015.

[2] c.h.Gu. Equations of mathematical physics. 2015.

[3] Tobias H Colding and William P Minicozzi. *A course in minimal surfaces*, volume 121. American Mathematical Soc., 2011.

[4] Klaus Ecker. *Regularity theory for mean curvature flow*, volume 57. Springer Science & Business Media, 2012.

[5] Jürgen Jost. *Partial differential equations*, volume 214. Springer Science & Business Media, 2012.

[6] John Willard Milnor. *Morse theory*. Number 51. Princeton university press, 1963.

[7] John Willard Milnor and David W Weaver. *Topology from the differentiable viewpoint*, volume 21. Princeton university press, 1997.

[8] Leon Simon. Introduction to geometric measure theory. *Tsinghua Lectures*, 2(2):3–1, 2014.

[9] s.l.Zhou. *Elliptic partial differential equations*. Peking University Press, 2005.

[10] Peter Topping. *Lectures on the Ricci flow*, volume 325. Cambridge University Press, 2006.

[11] y.b.Shen. *the First step to Global Differetial Geometry*. Higher Education Press, 2009.